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On inertial waves in a rotating fluid sphere

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Several new results are obtained for the classical problem of inertial waves in a rotating fluid sphere which was formulated by Poincaré more than a century ago. Explicit general analytical expressions for solutions of the problem are found in a rotating sphere for the first time. It is also discovered that there exists a special class of three-dimensional inertial waves that are nearly geostrophic and always travel slowly in the prograde direction. On the basis of the explicit general expression we are able to show that the internal viscous dissipation of all the inertial waves vanishes identically for a rotating fluid sphere. The result contrasts with the finite values obtained for the internal viscous dissipation for all other cases in which inertial waves have been studied.

1. Introduction

In rotating fluid systems with small viscosity and large rotation rates, fluid motions in the form of slowly decaying oscillations are found which are usually referred to as inertial waves or inertial oscillations (Greenspan 1968). Inertial waves can be sustained in different ways. Aldridge & Toomre (1969) used external time-dependent disturbances to excite axisymmetric spherical inertial oscillations in an experimental study of the problem for a sphere. Inertial waves are likely to be generated in the Earth's fluid outer core (Aldridge & Lumb 1987; Hide 1966). A subclass of the inertial waves in a rotating sphere which are nearly independent of the coordinate parallel to the axis of rotation can be excited and maintained by thermal convection when the Prandtl number of the fluid is sufficiently small (Zhang & Busse 1987; Zhang 1994, 1995; Ardes, Busse & Wicht 1996). In the context of magnetohydrodynamics, magnetically modified inertial waves can be obtained in the presence of an externally imposed magnetic field (Malkus 1967, 1968; Roberts & Loper 1979; Kerswell 1994; Zhang & Busse 1995).

There are usually two different sources of viscous dissipation in connection with the inertial waves in rotating systems. When a no-slip boundary condition is used, viscous dissipation occurs in the interior of the fluid, as well as in the Ekman boundary layer, with the latter being dominant (see, for example, Greenspan 1968). When the stress-free boundary condition is used, at most a weak Ekman boundary layer is realized and the internal viscous dissipation primarily determines in which way inertial waves can be excited and maintained in rotating fluid systems. For example, the Ekman boundary layers are not present in a rotating plane layer when the stress-free conditions are used (Zhang & Roberts 1997); it is solely the internal viscous dissipation associated with the three-dimensional integral

$$\int_V \boldsymbol{u}^* \cdot \nabla^2 \boldsymbol{u} \, \mathrm{d}V = \langle \boldsymbol{u}^* \cdot \nabla^2 \boldsymbol{u} \rangle \tag{1.1}$$

that plays a key role in determining the properties of convection-driven inertial waves. Here $\boldsymbol{u}$ is the velocity of a three-dimensional inertial wave, $\boldsymbol{u}^*$ is its complex conjugate and V denotes the volume of the fluid domain. We shall refer to (1.1) as the dissipation integral in this paper. In simple geometries such as a fluid layer (Zhang & Roberts 1997), it is straightforward to show that the dissipation integral is always non-zero and negative. In Appendix A, we show that the dissipation integral of an inertial wave in a rotating annulus, which has curvature and is widely used to mimic spherical geometry, is always negative. This is consistent with the anticipation that there is always finite viscous dissipation of the wave in the interior of the fluid. It is discovered, however, that the dissipation integral (1.1) vanishes identically in a rotating fluid sphere. In order to demonstrate that this peculiar feature is true for all inertial waves, we have obtained explicit general analytical expressions that describe all three-dimensional inertial waves in a rotating fluid sphere. We are thus able to show that the internal viscous dissipation for all inertial waves in a fluid sphere does indeed vanish.

In what follows we present a brief mathematical formulation and the relevant previous results in § 2. In § 3 we shall discuss a particular example of an inertial wave in a rotating fluid sphere for the purpose of illustration. In §§ 4 and 5 we study the general problem, with some concluding remarks given in § 6.

2. The inertial wave problem in a sphere

The problem of inertial waves and oscillation which describes the motion of an inviscid fluid in a rotating fluid sphere is a classical one. More than a century ago Poincaré derived the basic governing equation and Bryan obtained the general implicit solution in modified oblate spheroidal coordinates (Bryan 1889; Lyttleton 1953). A detailed account of earlier research results on this problem can be found in Greenspan (1968).

Consider a homogeneous fluid sphere of small viscosity ν that is rotating uniformly with a constant angular velocity Ω. It is convenient to take the radius of the sphere, r_o, as a characteristic length, and Ω^{-1} as a characteristic scale of time. The dimensionless linearized equations of motion and continuity are

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{k} \times \boldsymbol{u} = -\nabla P + [E\nabla^2 \boldsymbol{u} - \boldsymbol{f}], \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1b}$$

where $\boldsymbol{k}$ is the unit vector parallel to the axis of rotation and the Ekman number E is defined as

$$E = \frac{\nu}{\Omega r_o^2},$$

and is assumed to be a small parameter; $\boldsymbol{f}$ represents a small external force required to sustain the wave motion against the weak viscous dissipation.

When E is sufficiently small, the fluid motions can be described in leading order as inertial waves for which the terms $[E\nabla^2\boldsymbol{u} - \boldsymbol{f}]$ can be dropped from equation (2.1a). Neglecting the effect of an Ekman boundary layer, we can solve the problem defined by equations (2.1a, b) through the perturbation ansatz

$$\boldsymbol{u} = \boldsymbol{u}_0 + E\boldsymbol{u}_1 + \dots, \quad P = P_0 + EP_1 + \dots, \quad \boldsymbol{f} = E\boldsymbol{f}_0 + \dots . \qquad (2.2a, b, c)$$

The leading-order problem describing inertial waves is given by

$$\frac{\partial \boldsymbol{u}_0}{\partial t} + 2\boldsymbol{k} \times \boldsymbol{u}_0 = -\nabla P_0, \quad \nabla \cdot \boldsymbol{u}_0 = 0 \qquad (2.3a, b)$$

with the boundary condition

$$\boldsymbol{r} \cdot \boldsymbol{u}_0 = 0 \qquad (2.4)$$

on the surface of the fluid sphere. The details of the higher-order problem, which depends on the way in which the flow is driven or excited, do not concern us in this paper. Of importance, however, is the dissipation integral which is usually associated with the solvability condition

$$\mathrm{Re}[\langle \boldsymbol{u}_0^* \cdot \boldsymbol{f}_0 \rangle] = \mathrm{Re}[\langle \boldsymbol{u}_0^* \cdot \nabla^2 \boldsymbol{u}_0 \rangle], \qquad (2.5)$$

where $\langle\, \rangle$ denotes a three-dimensional integration over the entire sphere.

To determine the solution $\boldsymbol{u}_0$, we eliminate the pressure P_0 and rewrite equations (2.3a, b) in terms of the three velocity components in cylindrical coordinates (s, ϕ, z). Let

$$\boldsymbol{u}_0 = [U_s(s, z), U_\phi(s, z), U_z(s, z)]\, \mathrm{e}^{\mathrm{i}(m\phi + 2\sigma t)}. \qquad (2.6)$$

The vector equations (2.3a, b) can then be written as the three scalar equations

$$s\frac{\partial U_s}{\partial z} + \mathrm{i}\sigma s \frac{\partial U_\phi}{\partial z} + m\sigma U_z = 0, \qquad (2.7a)$$

$$\frac{\partial (sU_s)}{\partial s} + \mathrm{i}\sigma \frac{\partial (sU_\phi)}{\partial s} + m(\mathrm{i}U_\phi + \sigma U_s) = 0, \qquad (2.7b)$$

$$\frac{\partial (sU_s)}{\partial s} + \mathrm{i}mU_\phi + s\frac{\partial U_z}{\partial z} = 0, \qquad (2.7c)$$

with the boundary condition

$$sU_s + zU_z = 0 \quad \text{at} \quad s^2 + z^2 = 1, \qquad (2.8)$$

where m is the azimuthal wavenumber of the inertial wave.

Two different classes of solutions of equations (2.7a–c) can be distinguished: equatorially symmetric waves are characterized by the symmetry property

$$(U_\phi, U_s, U_z)(z) = (U_\phi, U_s, -U_z)(-z); \qquad (2.9a)$$

while equatorially antisymmetric waves obey the symmetry

$$(U_\phi, U_s, U_z)(z) = (-U_\phi, -U_s, U_z)(-z). \qquad (2.9b)$$

The mathematical expressions and analysis for both the parities are identical except for shifting some indexes by 1. In order to discuss and express the already complex mathematical problem with many indexes more clearly, we shall mainly focus on the equatorially symmetric waves (2.9a). The analysis for the equatorially antisymmetric waves (2.9b) is exactly parallel and will not provide any additional insight into the problem.

Implicit solutions for (2.7a–c) were first derived more than a century ago. The brief discussion below follows that given by Bryan (1889). Introducing the modified spheroidal coordinates X and Y

$$X = \frac{\sqrt{2}(1-\sigma^2)^{1/2}}{2}\left[\sqrt{d^2 - \frac{4\sigma^2 z^2}{(1-\sigma^2)^2}} - d\right]^{1/2}, \tag{2.10a}$$

$$Y = \frac{\sqrt{2}\sigma z}{(1-\sigma^2)^{1/2}}\left[\sqrt{d^2 - \frac{4\sigma^2 z^2}{(1-\sigma^2)^2}} - d\right]^{-1/2}, \tag{2.10b}$$

with

$$d = s^2 - \frac{(1+\sigma^2 z^2)}{(1-\sigma^2)},$$

we can express the velocity of the inertial wave in a rotating sphere compactly as

$$U_z = \mathrm{i}\frac{(1-\sigma^2)}{\sigma}\left(\frac{\mathrm{d}\Phi(X)}{\mathrm{d}X}\Phi(Y)X_z + \frac{\mathrm{d}\Phi(Y)}{\mathrm{d}Y}\Phi(X)Y_z\right), \tag{2.11a}$$

$$U_s = -\frac{\mathrm{i}}{s}\left(m\Phi(X)\Phi(Y) + \frac{\mathrm{d}\Phi(X)}{\mathrm{d}X}\Phi(Y)\sigma s X_s + \frac{\mathrm{d}\Phi(Y)}{\mathrm{d}Y}\Phi(X)\sigma s Y_s\right), \tag{2.11b}$$

$$U_\phi = \frac{1}{s}\left(m\sigma\Phi(X)\Phi(Y) + \frac{\mathrm{d}\Phi(X)}{\mathrm{d}X}\Phi(Y)sX_s + \frac{\mathrm{d}\Phi(Y)}{\mathrm{d}Y}\Phi(X)sY_s\right), \tag{2.11c}$$

where $\Phi(x)$ denotes an associated Legendre function of order m and

$$X_s = -\frac{sX(1-\sigma^2)}{X^2-Y^2}, \quad Y_s = \frac{sY(1-\sigma^2)}{X^2-Y^2},$$

$$X_z = -\frac{Y(1-X^2)\sigma}{X^2-Y^2}, \quad Y_z = \frac{X(1-Y^2)\sigma}{X^2-Y^2}.$$

Upon realizing that the complex transformation (2.10a,b) causes difficulties in the derivation of explicit solutions, Kudlick (1966) (see Greenspan 1968) obtained an improved implicit solution of the inertial waves in the form of a summation of polynomials:

$$U_z = \mathrm{i}\frac{(1-\sigma^2)}{\sigma}\frac{\partial F_N(s,z)}{\partial z}, \tag{2.12a}$$

$$U_s = -\mathrm{i}\left(\sigma\frac{\partial F_N(s,z)}{\partial s} + \frac{mF_N(s,z)}{s}\right), \tag{2.12b}$$

$$U_\phi = \frac{\partial F_N(s,z)}{\partial s} + \frac{m\sigma F_N(s,z)}{s}, \tag{2.12c}$$

where F_N is a double polynomial given by

$$F_N(s,z) = s^m \prod_{k=1}^{N}(a_k s^2 + b_k z^2 + c_k) \tag{2.13}$$

with

$$a_k = (1-\sigma^2)x_k, \quad b_k = \sigma^2(1-x_k), \quad c_k = x_k(x_k-1),$$

where $x_k,\ k = 1, 2, \ldots N$, are the N roots of the equation

$$\sum_{j=0}^{N} (-1)^j \frac{[2(2N+m-j)]!}{j![2(N-j)]!(2N+m-j)!}\, x_k^{N-j} = 0. \tag{2.14}$$

The half-frequency, σ, of an equatorially symmetric inertial wave (2.9*a*), corresponds to a root of the equation

$$W_{Nm}(\sigma_{Nmn}) = \sum_{j=0}^{N} (-1)^j \times \frac{[2(2N+m-j)]!}{j!(2N+m-j)![2(N-j)]!} \left[(m+2N-2j) - \frac{2(N-j)}{\sigma_{Nmn}}\right] \sigma_{Nmn}^{2(N-j)} = 0, \tag{2.15}$$

while σ for an equatorially antisymmetric inertial wave (2.9*b*) is given by

$$W_{Nm}(\sigma_{Nmn}) = \sum_{j=0}^{N} (-1)^j \frac{[2(2N+m+1-j)]!}{j!(2N+m+1-j)![2(N-j)+1]!} \times \left[(m+2N-2j+1) - \frac{2(N-j)+1}{\sigma_{Nmn}}\right] \sigma_{Nmn}^{2(N-j)+1} = 0. \tag{2.16}$$

Accordingly, we can use the following procedure to calculate an explicit solution for the inertial wave: (i) solve the equation (2.14) to obtain the N roots, $x_k,\ k = 1, 2 \ldots N$; (ii) use (2.13) to find the double polynomial $F_N(s, z)$, and (iii) obtain an explicit expression for the wave velocity by using (2.12). When N is small, the explicit solution can be readily derived by this procedure. $N = 0, 1$ has been discussed in various contexts (for example, Malkus 1967, 1968; Zhang 1993). When $N > 4$, however, $x_k,\ k = 1, 2 \ldots N$ cannot be written as an analytical expression and, hence, the coefficients of the polynomial $F_N(s, z)$ cannot be expressed explicitly. In consequence a general explicit solution cannot be obtained by this procedure for $N > 4$ based on equations (2.12)–(2.14).

3. An example: symmetric waves with $N = 2$

Before discussing the general solution for the inertial wave in a rotating sphere and showing that the internal viscous dissipation of all the inertial waves vanishes identically in a sphere, it is profitable to illustrate this unique and salient characteristic by examining a simple subclass $N = 2$, which can be readily derived from equations (2.12)–(2.14). We first obtain the two roots from quadratic equation (2.14) at $N = 2$:

$$x_1 = \frac{1}{(2m+7)}\left[3 + \left(\frac{12(m+2)}{(2m+5)}\right)^{1/2}\right], \quad x_2 = \frac{1}{(2m+7)}\left[3 - \left(\frac{12(m+2)}{(2m+5)}\right)^{1/2}\right], \tag{3.1a,b}$$

which are then substituted into equation (2.13) to yield

$$F_2(s, z) = s^m[C_1 z^4 + C_2 s^4 + C_3 z^2 + C_4 s^2 + C_5 s^2 z^2 + C_6] \tag{3.2}$$

where expressions for $C_J,\ j = 1, \ldots, 6$, which are a function of the wavenumber m and half-frequency σ, are given in Appendix B. Using equation (2.12), we obtain an explicit expression for the velocity of the inertial wave of the subclass $N = 2$ for a

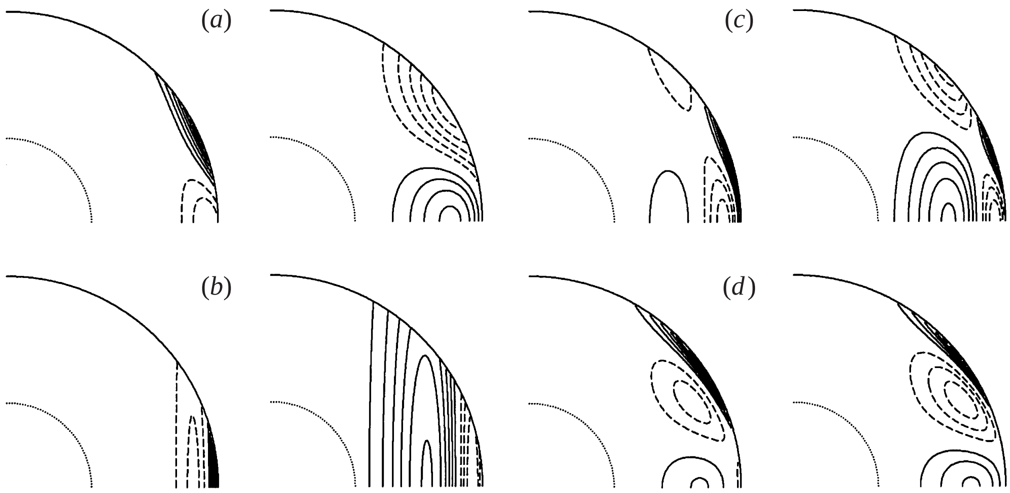

FIGURE 1. Contours of U_ϕ (on the left) and of the corresponding U_s (on the right) are displayed in a meridional plane ($z > 0$) for $N = 2$ with $m = 8$. The part for $z < 0$ is not shown because of symmetry (2.9*a*). There are four possible waves for the subclass: (*a*) $\omega = -0.8107$, (*b*) $\omega = -0.1450$, (*c*) $\omega = 0.5037$ and (*d*) $\omega = 1.1187$, where ω is the frequency of the wave and $\omega = 2\sigma$. The solid lines represent flow in the eastward direction for U_ϕ (away from the axis of rotation for U_s), while the dashed lines denote the westward direction.

given σ,

$$U_z = \mathrm{i}\frac{4(1-\sigma^2)}{\sigma}[C_1^z z^3 + C_2^z z s^2 + C_3^z z]s^m, \tag{3.3a}$$

$$U_s = -\mathrm{i}[C_1^s z^4 + C_2^s s^4 + C_3^s z^2 + C_4^s s^2 + C_5^s s^2 z^2 + C_6^s]s^{m-1}, \tag{3.3b}$$

$$U_\phi = [C_1^\phi z^4 + C_2^\phi s^4 + C_3^\phi z^2 + C_4^\phi s^2 + C_5^\phi s^2 z^2 + C_6^\phi]s^{m-1}. \tag{3.3c}$$

The coefficients in the above expressions are given in Appendix B.

Typically, for $m = 1$ the fluid motions are non-zero on the axis of rotation while they vanish there whenever $m > 1$. To display this characteristic difference and also to show the effect of the size of the wavenumber, we present two typical profiles of the waves, $m = 8$ in figure 1 and $m = 1$ in figure 2. For each wavenumber m, there are always four different inertial waves corresponding to the four values of σ for $N = 2$ (see Appendix B). As must be expected the nearly geostrophic waves shown in figures 1(*a*) and 2(*b*) are associated with the smallest $|\sigma|$ in this subclass. To show the possible effect of an inner sphere, we plot its position (the dotted line) with radius $0.35r_o$ in figures 1 and 2.

It is convenient to represent, as we have already done in § 2, a three-dimensional wave by employing the triple index notation, for example σ_{Nmn} or $\boldsymbol{u}_{Nmn}$, in which the wavenumber m always indicates the azimuthal scale of an inertial wave, N represents the degree of the possible complexity in the axial direction and n is related to the radial structure. While the wavenumber m can be any integer, N and n are related: the maximum value of n is determined by N in that $n = 1, 2, \ldots, 2N$ for symmetric modes.

Using (3.3), it is straightforward, but rather tedious, to show for the subclass $N = 2$ that

$$\int_V \boldsymbol{u}_{2mn}^* \cdot \nabla^2 \boldsymbol{u}_{2mn}\,\mathrm{d}V \equiv 0 \tag{3.4}$$

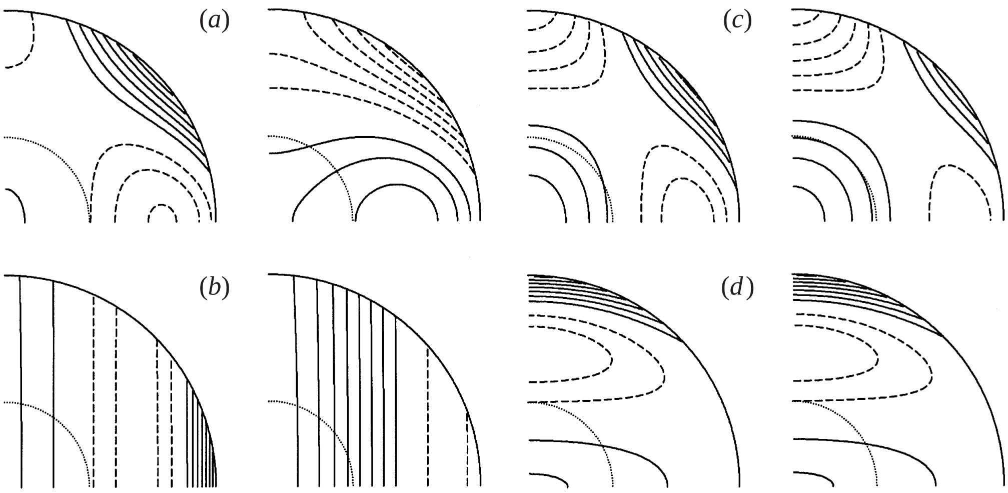

FIGURE 2. As figure 1 but with $m = 1$. Four possible waves for the subclass are: (*a*) $\omega = -1.1834$, (*b*) $\omega = -0.0682$, (*c*) $\omega = 1.0456$ and (*d*) $\omega = 1.8060$.

for any m or n and σ, whether σ satisfies or does not satisfy the eigenvalue equation (B1) given in Appendix B. This unexpected property was first noticed by Zhang (1994) for the subclass $N = 1$. But it was thought that it was a consequence of the particularly simple spatial structure for that subclass.

We have carried out the analysis for two more subclasses, $N = 3$ and $N = 4$, which have much more complex spatial structures, and the same property

$$\langle \boldsymbol{u}^*_{3mn} \cdot \nabla^2 \boldsymbol{u}_{3mn} \rangle = \langle \boldsymbol{u}^*_{4mn} \cdot \nabla^2 \boldsymbol{u}_{4mn} \rangle \equiv 0, \tag{3.5}$$

was found, indicating the possibility that it holds for all inertial waves in a sphere.

4. The new explicit solution

In order to show that equation (3.5) does indeed hold for all inertial waves in a sphere for $N > 4$, it is essential that explicit general expressions are obtained in order to extend the results (3.5) to $N > 4$. The process of derivation of the general solutions for (2.7) and (2.8) is very lengthy and cumbersome, and we only give a brief outline here. Substituting expressions (2.10*a, b*) and expressions for X_s, X_z, Y_s, Y_z into equations (2.11*a–c*) and making an appropriate reorganization, all the terms can be written in forms such as

$$X^2Y^2, \quad X^2 + Y^2, \quad X^4 + Y^4, \quad X^2Y^4 + X^4Y^2, \quad X^6 + Y^6.$$

They can be transformed into polynomials in terms of s, z and σ:

$$X^2Y^2 = \sigma^2 z^2, \quad X^2 + Y^2 = 1 - (1-\sigma^2)s^2 + \sigma^2 z^2,$$

$$X^4 + Y^4 = 1 - 2(1-\sigma^2)s^2 - 2(1-\sigma^2)\sigma^2 s^2 z^2 + (1-\sigma^2)^2 s^4 + \sigma^4 z^4,$$

$$X^4Y^2 + Y^4X^2 = \sigma^2 z^2 - 2\sigma^2(1-\sigma^2)s^2z^2 - 2(1-\sigma^2)\sigma^4 s^2 z^4 + \sigma^2(1-\sigma^2)^2 s^4 z^2 + \sigma^6 z^6,$$

$$\begin{aligned} X^6 + Y^6 = {} & 1 - 3(1-\sigma^2)s^2 + 3(1-\sigma^2)^2 s^4 - 3\sigma^2(1-\sigma^2)s^2z^2 - 3(1-\sigma^2)\sigma^4 s^2 z^4 \\ & + 3\sigma^2(1-\sigma^2)^2 s^4 z^2 - 3(1-\sigma^2)^3 s^6 + \sigma^6 z^6. \end{aligned}$$

By extending the above process, we can obtain the following explicit general expressions describing equatorially symmetric inertial waves, for all subclasses $N = 1, 2, 3, \ldots$:

$$U_z = \mathrm{i}\sum_{i=1}^{N}\sum_{j=0}^{N-i} C_{ijmN}\sigma_{Nmn}^{2i-1}(1-\sigma_{Nmn}^2)^j(2i)s^{m+2j}z^{2i-1}, \qquad (4.1a)$$

$$U_s = -\mathrm{i}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}\sigma_{Nmn}^{2i}(1-\sigma_{Nmn}^2)^{j-1}(m+m\sigma_{Nmn}+2j\sigma_{Nmn})s^{m+2j-1}z^{2i}, \qquad (4.1b)$$

$$U_\phi = \sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}\sigma_{Nmn}^{2i}(1-\sigma_{Nmn}^2)^{j-1}(m+m\sigma_{Nmn}+2j)s^{m+2j-1}z^{2i}, \qquad (4.1c)$$

where C_{ijmN} is defined as

$$C_{ijmN} = \frac{(-1)^{i+j}[2(m+N+i+j)-1]!!}{2^{j+1}(2i-1)!!(N-i-j)!i!j!(m+j)!}.$$

It can be readily shown that (4.1) satisfies (2.7a) since substitution of (4.1) into equation (2.7a) yields

$$s\frac{\partial U_s}{\partial z} + \mathrm{i}\sigma_{Nmn}s\frac{\partial U_\phi}{\partial z} + m\sigma_{Nmn}U_z \sim \sum_{i=1}^{N}\sum_{j=0}^{N-i} C_{ijmN}\sigma_{Nmn}^{2i}(1-\sigma_{Nmn}^2)^j(2i)s^{m+2j}z^{2i-1}$$
$$\times[m+(m\sigma_{Nmn}^2-m)(1-\sigma_{Nmn}^2)^{-1}] \equiv 0. \qquad (4.2)$$

In a similar way equation (2.7b) can be validated. For the equation of continuity, insertion of (4.1) into equation (2.7c) gives rise to

$$\frac{\partial(sU_s)}{\partial s} + \mathrm{i}mU_\phi + s\frac{\partial U_z}{\partial z} \sim \sum_{i=0}^{N-1}\sum_{j=0}^{N-i-1} \sigma_{Nmn}^{2i+1}(1-\sigma_{Nmn}^2)^j$$
$$\left[2(j+1)(m+j+1)C_{i(j+1)mN} - (i+1)(2i+1)C_{(i+1)jmN}\right] s^{m+2j} z^{2i} \equiv 0, \qquad (4.3)$$

because

$$2(j+1)(m+j+1)C_{i(j+1)mN} = (i+1)(2i+1)C_{(i+1)jmN}$$
$$= \frac{(-1)^{i+j+1}[2(m+N+i+j)+1]!!}{2^{j+1}(2i-1)!!(N-i-j)!i!j!(m+j)!}.$$

The boundary condition (2.8) is also satisfied since

$$zU_z+(1-z^2)^{1/2}\,U_s \sim W_{Nm}(\sigma_{Nmn})\left[\sum_{j=0}^{N}(-1)^j\frac{[2(2N+m-j)]!}{j!(2N+m-j)!(2N-j)!}z^{2N-2j}\right], \qquad (4.4)$$

which is zero provided that σ_{Nmn} is an eigenvalue given by equation (2.15). An intriguing feature is, however, that dissipation integrals like (3.5) vanish independently of the precise value of σ_{Nmn}, whether σ_{Nmn} satisfies (2.15) or not.

It is found that there exists a special class of the geostrophic inertial wave, i.e. where the wave motions are nearly independent of the axis of rotation, which is characterized by a small frequency. Since the inertial effect in equation (2.3a) is of secondary importance in the case of a slowly oscillatory wave, the fluid motions have

	$m = 1$		$m = 8$	
N	$(\omega_G)^{exact}$	$(\omega_G)^{approx}$	$(\omega_G)^{exact}$	$(\omega_G)^{approx}$
1	−0.17661	−0.17661	−0.25653	−0.25653
2	−0.06819	−0.06796	−0.14502	−0.14087
3	−0.03615	−0.03606	−0.09681	−0.09390
4	−0.02239	−0.02235	−0.07023	−0.06833
5	−0.01523	−0.01521	−0.05366	−0.05240
2	−0.01103	−0.01102	−0.04251	−0.04164
7	−0.00836	−0.00835	−0.03459	−0.03398
8	−0.00655	−0.00655	−0.02874	−0.02830
10	−0.00433	−0.00433	−0.02081	−0.02056

TABLE 1. Comparison between the frequencies given by (4.5) ($(\omega_G)^{approx}$) and the corresponding frequencies obtained from equation (2.15) ($(\omega_G)^{exact}$) for the nearly geostrophic waves.

to be nearly geostrophic. When the frequency of a geostrophic wave is sufficiently small,

$$\sigma_G = \sigma_{Nmn} \ll O\left[\frac{m}{N(m+N)}\right],$$

where $m > 0$, it can be readily shown that frequency ω_G for the geostrophic wave is given approximately by

$$\omega_G = 2\sigma_G = -\frac{2}{m+2}\left[\sqrt{1+\frac{m(m+2)}{N(2N+2m+1)}}-1\right]. \tag{4.5}$$

It follows that these nearly geostrophic inertial waves always travel slowly in the eastward direction. Table 1 gives several examples for the frequencies of the geostrophic waves which are calculated using (2.15) and (4.5). It is interesting to note that the approximate σ_G is exactly the same as that given by (2.15) when $N = 1$.

The explicit general expressions for all equatorially antisymmetric inertial waves are given by

$$U_z = \mathrm{i}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}\sigma_{Nmn}^{2i-1}(1-\sigma_{Nmn}^2)^j(2i+1)s^{m+2j}z^{2i}, \tag{4.6a}$$

$$U_s = -\mathrm{i}\sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}\sigma_{Nmn}^{2i}(1-\sigma_{Nmn}^2)^{j-1}(m+m\sigma_{Nmn}+2j\sigma_{Nmn})s^{m+2j-1}z^{2i+1}, \tag{4.6b}$$

$$U_\phi = \sum_{i=0}^{N}\sum_{j=0}^{N-i} C_{ijmN}\sigma_{Nmn}^{2i}(1-\sigma_{Nmn}^2)^{j-1}(m+m\sigma_{Nmn}+2j)s^{m+2j-1}z^{2i+1}, \tag{4.6c}$$

where C_{ijmN} is defined by

$$C_{ijmN} = \frac{(-1)^{i+j}[2(m+N+i+j)+1]!!}{2^{j+1}(2i+1)!!(N-i-j)!i!j!(m+j)!}.$$

Expressions (4.6) are valid for $N = 0, 1, 2, 3\ldots$ and possess the symmetry property (2.9*b*). Evidently, there are no nearly geostrophic modes for this particular symmetry because the parity does not permit it.

5. The vanishing of the general dissipation integral

After the explicit general expressions (4.1) and (4.6) for the complete set of inertial waves had been obtained, a great effort was made to show that the dissipation integral vanishes for all inertial waves. Here only the analysis for equatorially symmetric waves (4.1) will be demonstrated. While the relevant integrations over the sphere are not difficult, the resulting expressions involve rather complex sums with four different indexes:

$$\langle \boldsymbol{u}^*_{Nmn} \cdot \nabla^2 \boldsymbol{u}_{Nmn} \rangle \sim \left[4(1+\sigma^2_{Nmn})S_{1N} + 8m\sigma_{Nmn}S_{2N} + \frac{S_{3N}}{4\sigma^2_{Nmn}} \right], \tag{5.1}$$

where

$$S_{1N} \sim \sum_{i=0}^{N}\sum_{k=1}^{N}\sum_{j=0}^{N-i}\sum_{l=0}^{N-k}(-1)^{i+j+k+l}\sigma_{Nmn}^{2(i+k)}(1-\sigma^2_{Nmn})^{j+l} \times \frac{[m^2+(m+2j)(m+2l)][2(m+N+i+j)-1]!!}{[2(l+k+i+j+m)-1]!!} \times \frac{[2(m+N+k+l)-1]!!}{(2i-1)!!(N-i-j)!i!j!(m+j)!(k-1)!l!}\,\frac{(2i+2k-3)!!(l+j+m-1)!}{(2k-3)!!(l+m)!(N-k-l)!}, \tag{5.2a}$$

$$S_{2N} \sim \sum_{i=0}^{N}\sum_{k=1}^{N}\sum_{j=0}^{N-i}\sum_{l=0}^{N-k}(-1)^{i+j+k+l}\sigma_{Nmn}^{2(i+k)}(1-\sigma^2_{Nmn})^{j+l} \times \frac{[2(m+N+i+j)-1]!!}{[2(l+k+i+j+m)-1]!!} \times \frac{[2(m+N+k+l)-1]!!}{(2i-1)!!(N-i-j)!i!j!(m+j)!(k-1)!l!}\,\frac{(2i+2k-3)!!(l+j+m)!}{(2k-3)!!(l+m)!(N-k-l)!}, \tag{5.2b}$$

$$S_{3N} \sim \sum_{i=1}^{N}\sum_{k=2}^{N}\sum_{j=0}^{N-i}\sum_{l=0}^{N-k}(-1)^{i+j+k+l}\sigma_{Nmn}^{2(i+k)}(1-\sigma^2_{Nmn})^{j+l} \times \frac{[2(m+N+i+j)-1]!!}{[2(l+k+i+j+m)-1]!!} \times \frac{[2(m+N+k+l)-1]!!}{(2i-1)!!(N-i-j)!(i-1)!j!(m+j)!(k-2)!l!}\,\frac{(2i+2k-5)!!(l+j+m)!}{(2k-3)!!(l+m)!(N-k-l)!}, \tag{5.2c}$$

where $N > 1$. We find that

$$S_{1N} = S_{2N} = S_{3N} \equiv 0,$$

for any m, n, σ_{Nmn} and N, and hence that

$$\langle \boldsymbol{u}^*_{Nmn} \cdot \nabla^2 \boldsymbol{u}_{Nmn} \rangle = 0 \tag{5.3}$$

holds for all the inertial waves in a rotating fluid sphere.

It is evident that the indices, i, j, k, l in equations (5.2a–c) are intimately entangled, which makes it difficult to reduce their number by carrying out the summation over individual indices. Though there may be even simpler ways to prove equation (5.3), here we present a way that seems to be the most straightforward one. As an example we derive the proof for $S_{2N} \equiv 0$. The method and idea for proving $S_{1N} \equiv 0$ and $S_{3N} \equiv 0$ are almost identical.

The major task is to disentangle the indices in the summations. An effective way is to introduce two additional indices, say α and β, by considering a new sum with six indices:

$$S_{2N}^{M} = \sum_{\alpha=0}^{M}\sum_{\beta=0}^{M-\alpha} Z_{\alpha,\beta}^{M} \sum_{i=0}^{N-M}\sum_{k=1}^{N-M}\sum_{j=0}^{N-i-M}\sum_{l=0}^{N-k-M} (-1)^{i+j+k+l}\sigma_{Nmn}^{2(i+k+2\alpha)}(1-\sigma_{Nmn}^{2})^{j+l+2\beta}$$
$$\times \frac{[2(m+N+i+j+\alpha+\beta)-1]!![2(m+N+k+l+\alpha+\beta)-1]!!(l+j+m+\beta)!}{i![2(i+\alpha)-1]!!j!(m+j+\beta)!(N-i-j-M)!(k-1)!}$$
$$\times \frac{[2(i+k+\alpha)-3]!!}{[2(k+\alpha)-3)]!!l!(l+m+\beta)!(N-k-l-M)![2(\gamma+\alpha+\beta+M)-1]!!}, \qquad (5.4)$$

where $\gamma = (l+k+i+j+m)$ and the coefficients $Z_{i,j}$ are defined as

$$Z_{0,0}^{0} = 1, \quad Z_{i,0}^{M+1} = (-1)^{M+1-i}\frac{(M+1)!}{(M+1-i)!i!}2^{M+1}, \qquad (5.5a,b)$$

$$Z_{0,i}^{M+1} = (-2)^{M+1-i}\frac{(M+1)!}{(M+1-i)!i!}, \quad Z_{i,M+1-i}^{M+1} = 2^{i}\frac{(M+1)!}{(M+1-i)!i!}, \qquad (5.5c,d)$$

and

$$Z_{i,j}^{M+1} = -2Z_{i,j}^{M} + 2Z_{i-1,j}^{M} + Z_{i,j-1}^{M}, \quad 1 \leqslant i \leqslant (M-1), \quad 1 \leqslant j \leqslant (M-i). \qquad (5.5e)$$

In fact the precise values of the coefficients $Z_{i,j}^{M}$ are not important because they are not required in our analysis. Clearly we have

$$S_{2N} = S_{2N}^{0}. \qquad (5.6)$$

At a first glance, the sum (5.4) involving the six entangled indices is much more complex than (5.2b). The idea is to find a recurrence relation between S_{2N}^{j} and S_{2N}^{j+1} instead of direct evaluation of (5.4). The hope is that the twisted indices can be decoupled and the relevant summations can readily be carried out once the new index M is sufficiently large. As shown in Appendix C, such a recurrence relation does exist and the decoupling of the indices does indeed happen when $M = N-1$. At $M = N-1$ the tangled indices can be decoupled and the expression (5.4) can readily be worked out explicitly:

$$S_{2N}^{N-1} = -\sum_{\alpha=0}^{N-1}\sum_{\beta}^{N-1-\alpha} Z_{\alpha,\beta}^{N-1}\frac{[2(m+N+\alpha+\beta)+1]!!}{(2\alpha-1)!!(m+\beta)!}$$
$$\times \left[\sum_{i=0}^{1}\sum_{j=0}^{1-i}\frac{(-1)^{i+j}\sigma_{Nmn}^{2(i+2\alpha+1)}(1-\sigma_{Nmn}^{2})^{j+2\beta}}{i!j!(1-i-j)!}\right] \qquad (5.7)$$

which is identically zero because

$$\sum_{i=0}^{1}\sum_{j=0}^{1-i}\frac{(-1)^{i+j}\sigma_{Nmn}^{2i}(1-\sigma_{Nmn}^{2})^{j}}{i!j!(1-i-j)!} \equiv 0. \qquad (5.8)$$

This implies, by the recurrence relation, that $S_{2N} \equiv 0$. In a similar way we can show that $S_{1N} = 0$ and $S_{3N} = 0$.

It is of importance to note that the internal dissipation integral vanishes for all inertial waves regardless of the spatial scale of a wave. When an inertial wave has a large spatial scale with the wavenumber $O(1)$, the viscous dissipation in the Ekman boundary layer with non-slip boundary condition is $O(E^{1/2})$ with $E \ll 1$, which is much larger than the possible internal dissipation $O(E)$. In this case, we may neglect the internal dissipation, at leading order anyway, either in a sphere or a plane layer, and the property (5.3) is not physically important. When an inertial wave has a small spatial scale, say, with a large wavenumber $O(E^{-1/3})$, the internal dissipation $O(E^{1/3})$ for a plane layer would be much larger than the viscous dissipation in the Ekman boundary layer. However, the internal dissipation with $m = O(E^{-1/3})$ for a rotating sphere is still identically zero. In this case, the property (5.3) is also physically important. It follows that the inertial wave in a rotating sphere is always controlled by the Ekman boundary layer, which may be weak such as in the case of stress-free boundary conditions.

6. Concluding remarks

The vanishing of the dissipation integral (1.1) for all inertial oscillations of a rotating sphere may suggest that it is caused by a simple relationship. We have been unable to find such a simple reason and we thus had to revert to the analysis in § 5 in order to provide a proof of the property. The fact that the dissipation integral of inertial waves is finite in all other geometric configurations that have been studied indicates that the vanishing of the dissipation integral is associated with the unique geometric properties of the sphere. For a general three-dimensional flow, $\boldsymbol{u} = (u_r, u_\theta, u_\phi)$, in a sphere, the dissipation integral (1.1) can be written as

$$\int_V \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} \, \mathrm{d}V = -\int_V |\nabla \times \boldsymbol{u}|^2 \, \mathrm{d}V + 2\int_S (u_\theta^2 + u_\phi^2) \, \mathrm{d}S + \int_S \left[u_\theta \frac{\partial (u_\theta / r)}{\partial r} + u_\phi \frac{\partial (u_\phi / r)}{\partial r} \right] \mathrm{d}S, \tag{6.1}$$

where S represents the surface of the sphere and the two surface integrals are associated with the boundary condition of the flow. If $\boldsymbol{u}$ satisfies a non-slip boundary condition, the two surface integrals in (6.1) vanish; if $\boldsymbol{u}$ satisfies a stress-free boundary condition, the second surface integral in (6.1) is then zero. However, the inertial wave solution given by (4.1) or (4.6) does not satisfy either the rigid or stress-free boundary condition.

An interesting question is whether the internal dissipation integral vanishes in a rotating spherical shell with a small inner sphere or in a rotating spheroid. A great effort has been made by us to study analytically the same problem in a rotating spherical shell. So far we have been unable to find exact solutions that satisfy equations (2.7*a*–*c*) as well as the boundary condition at both the inner and outer surfaces of the shell. Numerical solutions for inertial waves in a rotating spherical shell (for example, Hollerbach & Kerswell 1995; Rieutord & Valdettaro 1997) indicated that the solution of an inertial wave in a rotating spherical shell may be quite different from that in a sphere, at least for some modes. But it is not clear how and why the effect of a small inner sphere can change an inertial wave such as that shown in figure 1. Our study of the problem of inertial waves in a spheroid is currently underway. The preliminary results in connection with some simple solutions indicate that the dissipation integral does not vanish in general for a spheroid.

The explicit general solution for the inertial waves in a rotating sphere is likely to be the most useful result of this paper. The availability of explicit expressions opens new possibilities for studying problems of fluid dynamics such as thermal convection in rotating spherical systems or motion in the presence of an azimuthal magnetic field as assumed by Malkus (1967). The explicit solution (4.1) may be used as a basis to express an arbitrary velocity distribution in a sphere. In comparison to spherical harmonics the inertial wave eigenfunctions are more natural in some situations and there are numerical advantages in using them. The relevant research is now underway.

An important finding, which may be significant for the dynamics of the Earth's core, is the three-dimensional, nearly geostrophic slow waves. When N in equation (4.5) is sufficiently large, they represent slowly travelling, nearly two-dimensional columnar Rossby-type waves. These columnar Rossby waves resemble the columnar convection modes found by Busse (1970) (see also Roberts 1968) and in fact the relationship between Rossby waves and columnar convection becomes an exact correspondence in the limit of small Prandtl number in the case of the rotating cylindrical annulus (Busse 1986). In the case of the sphere slow inertial waves lack the phase shift in the azimuthal direction and thus cannot be directly related to the strongly spiralling columnar convection (Zhang 1992). On the other hand, a perfect correspondence between convection and inertial oscillations has already been pointed out by Zhang (1994, 1995) for equatorially attached convection at low Prandtl numbers.

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Appendix A

Consider the problem of inertial waves in a cylindrical annulus of height H with the inner and outer radii at s_i and s_o, which rotates about the symmetry axis with a uniform angular velocity Ω. It is convenient to use the height H as a characteristic length and Ω^{-1} as a characteristic scale of time. The inertial wave with the frequency $\omega = 2\sigma$ is bounded $|\sigma| < 1$ (Greenspan 1968). It can be readily shown that σ is given by the solution of the equation

$$\left[s_o K_{m-1}(\xi_o) + \frac{m(1-\sigma)^{1/2}}{n\pi(1+\sigma)^{1/2}} K_m(\xi_o)\right]\left[s_i J_{m-1}(\xi_i) + \frac{m(1-\sigma)^{1/2}}{n\pi(1+\sigma)^{1/2}} J_m(\xi_i)\right] - \left[s_o J_{m-1}(\xi_o) + \frac{m(1-\sigma)^{1/2}}{n\pi(1+\sigma)^{1/2}} J_m(\xi_o)\right]\left[s_i K_{m-1}(\xi_i) + \frac{m(1-\sigma)^{1/2}}{n\pi(1+\sigma)^{1/2}} K_m(\xi_i)\right] = 0, \tag{A 1}$$

where

$$\xi_i = \frac{n\pi}{\sigma}(1-\sigma^2)^{1/2} s_i, \quad \xi_o = \frac{n\pi}{\sigma}(1-\sigma^2)^{1/2} s_o, \tag{A 2}$$

and J_m and K_m are the Bessel function of the first and second kind, m denotes the azimuthal wavenumber and n represents the number of zeros of the wave motions along the axis of rotation. The velocity of the inertial waves can be explicitly written as

$$u_z = -\mathrm{i}\frac{(1-\sigma^2)n\pi}{\sigma} f_{mn}(s) \sin(n\pi z)\, \mathrm{e}^{\mathrm{i}(m\phi+2\sigma t)}, \tag{A 3a}$$

$$u_s = -\mathrm{i}\left[\sigma\frac{\mathrm{d}f_{mn}}{\mathrm{d}s} + \frac{mf_{mn}}{s}\right]\cos(n\pi z)\,\mathrm{e}^{\mathrm{i}(m\phi+2\sigma t)}, \tag{A 3b}$$

$$u_\phi = \left[\frac{\mathrm{d}f_{mn}}{\mathrm{d}s} + \frac{m\sigma f_{mn}}{s}\right]\cos(n\pi z)\,\mathrm{e}^{\mathrm{i}(m\phi+2\sigma t)}, \tag{A 3c}$$

where $\mathrm{i} = \sqrt{-1}$, $\boldsymbol{u} = (u_s, u_\phi, u_z)$ represents the velocity field in cylindrical coordinates (s, ϕ, z) and

$$f_{mn}(s) = [m(1-\sigma)^{1/2} K_m(\xi_o) + n\pi s_o(1+\sigma)^{1/2} K_{m-1}(\xi_o)]J_m(\xi) - [m(1-\sigma)^{1/2} J_m(\xi_o) + n\pi s_o(1+\sigma)^{1/2} J_{m-1}(\xi_o)]K_m(\xi) \tag{A 4}$$

with

$$\xi = \frac{n\pi}{\sigma}(1-\sigma^2)^{1/2}s.$$

For any σ obtained from (A1), we can use (A3) to find the dissipation integral

$$\langle \boldsymbol{u}^*\nabla^2\boldsymbol{u}\rangle \sim -\int_{s_i}^{s_o}\left[\frac{(n\pi(1-\sigma)f_{mn})^2}{\sigma^2} + \left(\sigma\frac{\mathrm{d}f_{mn}}{\mathrm{d}s} + \frac{mf_{mn}}{s}\right)^2 + \left(\frac{\mathrm{d}f_{mn}}{\mathrm{d}s} + \frac{m\sigma f_{mn}}{s}\right)^2\right]s\,\mathrm{d}s, \tag{A 5}$$

which is always non-zero and negative. If a no-slip boundary is assumed, the dissipation within the Ekman boundary layers would dominate; if the boundary condition is assumed to be stress-free, there will be no Ekman boundary layers on the ends of the annulus and the viscous dissipation in the weak Ekman layers at the walls and in the interior of the fluid would be equally important. In a similar way we can show a non-zero and negative dissipation integral for a rotating plane layer, cylinder and box.

Appendix B

In equation (3.2) coefficients C_j, $j = 1, \ldots, 6$, are a function of σ and m and are given by

$$C_1 = \frac{4\sigma^4}{9}(2m+5)(2m+7)(m+1)(m+2), \quad C_2 = \frac{(1-\sigma^2)^2}{3}(2m+5)(2m+7),$$

$$C_3 = -\frac{8\sigma^2}{3}(2m+5)(m+1)(m+2), \quad C_4 = -\frac{4(1-\sigma^2)}{3}(2m+5)(m+2),$$

$$C_5 = \frac{4\sigma^2(1-\sigma^2)}{3}(2m+5)(2m+7)(m+2), \quad C_6 = \frac{4}{3}(m+1)(m+2).$$

The coefficients in equation (3.3) are

$$C_1^z = \frac{2\sigma^4}{3}(2m+5)(2m+7), \quad C_2^z = -2\sigma^2(2m+5), \quad C_3^z = \frac{\sigma^2(1-\sigma^2)(2m+5)(2m+7)}{(m+1)},$$

$$C_1^s = C_1^\phi = \frac{2m\sigma^4}{3}(1+\sigma)(2m+5)(2m+7), \quad C_3^s = C_3^\phi = -4m\sigma^2(1+\sigma)(2m+5),$$

$$C_2^s = \frac{(1-\sigma^2)^2\,(2m+5)(2m+7)(m\sigma+4\sigma+m)}{2(m+1)(m+2)},$$

$$C_2^{\phi} = \frac{(1-\sigma^2)^2\,(2m+5)(2m+7)(m\sigma+4+m)}{2(m+1)(m+2)},$$

$$C_4^{s} = -\frac{2(1-\sigma^2)(2m+5)(m\sigma+2\sigma+m)}{(m+1)}, \quad C_4^{\phi} = -\frac{2(1-\sigma^2)(2m+5)(m\sigma+2+m)}{(m+1)},$$

$$C_5^{s} = \frac{2\sigma^2(1-\sigma^2)(2m+5)(2m+7)(m\sigma+2\sigma+m)}{(m+1)},$$

$$C_5^{\phi} = \frac{2\sigma^2(1-\sigma^2)(2m+5)(2m+7)(m\sigma+2+m)}{(m+1)},$$

$$C_6^{s} = C_6^{\phi} = 2m(1+\sigma).$$

For any wavenumber m, there are four different waves with the half-frequency σ given by

$$\sigma_1 = \frac{1}{m+4} - \tfrac{1}{2}(\sqrt{\beta_1}+\sqrt{\beta_2}+\sqrt{\beta_3}), \quad \sigma_2 = \frac{1}{m+4} - \tfrac{1}{2}(\sqrt{\beta_1}-\sqrt{\beta_2}-\sqrt{\beta_3}), \tag{B 1a,b}$$

$$\sigma_3 = \frac{1}{m+4} + \tfrac{1}{2}(\sqrt{\beta_1}-\sqrt{\beta_2}+\sqrt{\beta_3}), \quad \sigma_4 = \frac{1}{m+4} + \tfrac{1}{2}(\sqrt{\beta_1}+\sqrt{\beta_2}-\sqrt{\beta_3}), \tag{B 1c,d}$$

where

$$\beta_j = \frac{4(m+3)(m+5)}{(2m+7)(m+4)^2} - \frac{8}{2m+7}\left(\frac{(m+2)(m+5)}{(2m+5)(m+4)}\right)^{1/2}$$
$$\times \cos\left[\frac{1}{3}\cos^{-1}\left\{-\frac{1}{2}\left(\frac{(m+5)(2m+5)}{(m+2)(m+4)}\right)^{1/2}\right\} + \frac{2(j-1)\pi}{3}\right], \quad j=1,2,3. \tag{B 2}$$

Appendix C

To obtain a recurrence relation, we first notice that (5.4) can be decomposed into the following four different summations:

$$S_{2N}^{M} = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4,$$

where

$$\Lambda_1 = \sum_{\alpha=0}^{M}\sum_{\beta=0}^{M-\alpha} Z_{\alpha,\beta}^{M} \sum_{i=0}^{N-M-1}\sum_{k=1}^{N-M-1}\sum_{j=0}^{N-i-M-1}\sum_{l=0}^{N-k-M-1}$$
$$(-1)^{i+j+k+l}\,\sigma_{Nmn}^{2(i+k+2\alpha+2)}(1-\sigma_{Nmn}^2)^{j+l+2\beta}$$
$$\times\frac{2[2(m+N+i+j+\alpha+\beta)+1]!![2(m+N+k+l+\alpha+\beta)+1]!!(l+j+m+\beta)!}{i![2(i+\alpha)+1)!!j!(m+j+\beta)!(N-i-j-M-1)!(k-1)!}$$
$$\times\frac{[2(i+k+\alpha)-1]!!}{[2(k+\alpha)-1)]!!l!(l+m+\beta)!(N-k-l-M-1)![2(\gamma+\alpha+\beta+M)+3]!!}, \tag{C 1a}$$

$$\Lambda_2 = -\sum_{\alpha=0}^{M}\sum_{\beta=0}^{M-\alpha} Z_{\alpha,\beta}^{M} \sum_{i=0}^{N-M-1}\sum_{k=1}^{N-M-1}\sum_{j=0}^{N-i-M-1}\sum_{l=0}^{N-k-M-1} (-1)^{i+j+k+l}\,\sigma_{Nmn}^{2(i+k+2\alpha+1)}(1-\sigma_{Nmn}^2)^{j+l+2\beta+1}$$
$$\times\frac{[2(m+N+i+j+\alpha+\beta)+1]!![2(m+N+k+l+\alpha+\beta)+1]!!(l+j+m+\beta+1)!}{i![2(i+\alpha)-1)!!j!(m+j+\beta+1)!(N-i-j-M-1)!(k-1)!}$$
$$\times\frac{[2(i+k+\alpha)-3]!!}{[2(k+\alpha)-3)]!!l!(l+m+\beta+1)!(N-k-l-M-1)![2(\gamma+\alpha+\beta+M)+3]!!}, \tag{C 1b}$$

$$\Lambda_3 = -\sum_{\alpha=0}^{M}\sum_{\beta=0}^{M-\alpha} Z_{\alpha,\beta}^{M} \sum_{i=0}^{N-M-1}\sum_{k=1}^{N-M-1}\sum_{j=0}^{N-i-M-1}\sum_{l=0}^{N-k-M-1} (-1)^{i+j+k+l}\,\sigma_{Nmn}^{2(i+k+2\alpha)}(1-\sigma_{Nmn}^2)^{j+l+2\beta}$$
$$\times\frac{2[2(m+N+i+j+\alpha+\beta)-1]!![2(m+N+k+l+\alpha+\beta)-1]!!(l+j+m+\beta)!}{i![2(i+\alpha)-1)!!j!(m+j+\beta)!(N-i-j-M-1)!(k-1)!}$$
$$\times\frac{[2(i+k+\alpha)-3]!!}{[2(k+\alpha)-3)]!!l!(l+m+\beta)!(N-k-l-M-1)![2(\gamma+\alpha+\beta+M)+1]!!}, \tag{C 1c}$$

$$\Lambda_4 = \sum_{\alpha=0}^{M}\sum_{\beta=0}^{M-\alpha} Z_{\alpha,\beta}^{M} \sum_{i=0}^{N-M-1}\sum_{k=1}^{N-M-1}\sum_{j=0}^{N-i-M-1}\sum_{l=0}^{N-k-M-1} (-1)^{i+j+k+l}\,\sigma_{Nmn}^{2(i+k+2\alpha)}(1-\sigma_{Nmn}^2)^{j+l+2\beta+1}$$
$$\times\frac{[2(m+N+i+j+\alpha+\beta)+1]!![2(m+N+k+l+\alpha+\beta)+1]!!(l+j+m+\beta+1)!}{i![2(i+\alpha)-1)!!j!(m+j+\beta+1)!(N-i-j-M-1)!(k-1)!}$$
$$\times\frac{[2(i+k+\alpha)-3]!!}{[2(k+\alpha)-3)]!!l!(l+m+\beta+1)!(N-k-l-M-1)![2(\gamma+\alpha+\beta+M)+3]!!}. \tag{C 1d}$$

Secondly, we notice that, by shifting α in Λ_1 by 1, and by combining Λ_2 and Λ_4 and then shifting β by 1, we find the following relationship between S_{2N}^{M} and S_{2N}^{M+1}:

$$S_{2N}^{M} = \sum_{\alpha=0}^{(M+1)}\sum_{\beta=0}^{(M+1)-\alpha} Z_{\alpha,\beta}^{(M+1)} \sum_{i=0}^{N-(M+1)}\sum_{k=1}^{N-(M+1)}\sum_{j=0}^{N-i-(M+1)}\sum_{l=0}^{N-k-(M+1)} (-1)^{i+j+k+l}\,\sigma_{Nmn}^{2(i+k+2\alpha)}(1-\sigma_{Nmn}^2)^{j+l+2\beta}$$
$$\times\frac{[2(m+N+i+j+\alpha+\beta)-1]!![2(m+N+k+l+\alpha+\beta)-1]!!(l+j+m+\beta)!}{i![2(i+\alpha)-1)!!j!(m+j+\beta)![N-i-j-(M+1)]!(k-1)!}$$
$$\times\frac{[2(i+k+\alpha)-3]!!}{[2(k+\alpha)-3)]!!l!(l+m+\beta+1)![N-k-l-(M+1)]![2(\gamma+\alpha+\beta+(M+1))-1]!!}. \tag{C 2}$$

In other words, we have shown that there is a general recurrence relation for $N \geqslant 2$,

$$S_{2N}^{M} = \frac{S_{2N}^{M+1}}{N-M}, \tag{C 3}$$

which gives rise to

$$S_{2N} = S_{2N}^{0} = \frac{S_{2N}^{1}}{N} = \frac{S_{2N}^{2}}{N(N-1)} = \cdots = \frac{S_{2N}^{N-1}}{N!}. \tag{C 4}$$

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